Int. J. Heat Mass Transfer. Vol. 32, No. 10, pp. 1917–1926, 1989
Printed in Great Britain

0017–9310/89 $3.00+0.00

Temperature distribution in a plate with temperature-dependent thermal conductivity and internal heat generation

H. C. ÜNAL†

MT-TNO, P.O. Box 342, 7300 AH Apeldoorn, The Netherlands

(*Received* 15 *December* 1988)

Abstract—The analytic solution of the differential equation of the steady-state one-dimensional temperature distribution in a plate with linearly temperature-dependent thermal conductivity and internal heat generation is presented. It is shown that the solution of this differential equation with a numerical method and its analytic solution with the assumption that the thermal conductivity of the material of the plate is constant while it depends on the temperature of the plate linearly are trivial for some conditions, which may be important for practical applications. Simple criteria are given to avoid these trivial solutions.

INTRODUCTION

THE AIM of this paper is to present an exact analytic solution of the differential equation for the steady-state one-dimensional temperature distribution in a plate with linearly temperature-dependent thermal conductivity and internal heat generation. This solution is also of practical importance for the determination of exothermal and endothermal chemical reactions, heating with electric high frequency power and diffusion.

The foregoing equation is a second-order non-linear differential equation, and to the author's knowledge the exact solution of this differential equation is not available in the literature. Jakob [1] gave the steady-state one-dimensional temperature distribution in a plate with linearly temperature-dependent internal heat generation. Aziz [2] presented an approximate perturbation solution for a convective fin with constant internal heat generation and linearly temperature-dependent thermal conductivity. The differential equation dealt with herein is in fact identical to that solved by Aziz with an approximate perturbation method, provided that the appropriate changes in the notation, assumptions and boundary conditions are taken into account.

For some practical applications, the variation of the thermal conductivity with temperature may be of secondary importance and the foregoing differential equation can be solved assuming that the thermal conductivity of the material of a plate is constant. But this solution is trivial beyond some combinations of the known value of the base temperature of the plate (i.e. a boundary value) and the known values of the coefficients for the linear variations of the thermal conductivity and internal heat generation if the variation of the thermal conductivity with temperature is of secondary importance but linear, as will be demonstrated later. This seems in fact a foregone conclusion since it is dealt with as a physical phenomenon and the assumption of a constant thermal conductivity alters the nature of this physical phenomenon. Furthermore the solution of the differential equation dealt with herein with a numerical method yields also trivial solutions beyond the aforesaid combinations. These combinations may be of practical significance, as will be illustrated with a few examples later.

The contents of this paper are outlined below. First the differential equation of the steady-state one-dimensional temperature distribution in a plate with linearly temperature-dependent thermal conductivity and internal heat generation is solved, the solution being an implicit function of temperature and including Legendre's normal elliptic integrals of the first and the second kinds. Thereafter the determination of the temperature in a plate at a given location is explained. Finally it is shown that for a given plate the solution of the foregoing differential equation with the assumption that the thermal conductivity of the material of the plate is constant while this thermal conductivity is a linear function of the temperature of the plate and its solution with a numerical method are trivial for conditions which may be important for practical applications. In order to avoid these trivial solutions, criteria are presented.

The use of the above mentioned elliptic integrals is not common in heat transfer engineering. Therefore, these integrals are sufficiently treated in the Appendix so that a practising engineer can use the results of this study.

† Present address: Bolerostr. 11, 7323 KC Apeldoorn, The Netherlands.

NOMENCLATURE

A	constant defined in the text
a–d	constants defined in the text
C	constant of integration
$E(\varphi/\alpha)$	Legendre's normal elliptic integral of the second kind
$F(\varphi/\alpha)$	Legendre's normal elliptic integral of the first kind
h	dimensionless coefficient for the linear variation of thermal conductivity
J_1–J_4	functions defined in the Appendix
K	thermal conductivity [W m^{-1} K^{-1}]
k	constant defined in the text
KU	constant defined in the Appendix
K_1	constant defined in the Appendix
n	dimensional coefficient for the linear variation of thermal conductivity [K^{-1}]
P	variable defined in the text
Q	internal heat generation [W m^{-3}]
r	constant defined in the text
S	constant defined in the text
s	dimensionless coefficient for the linear variation of internal heat generation
T	dimensionless temperature
t	temperature [K]
U	thickness or half of the thickness of a plate [m]
W–W_{11}	constants or functions defined in the text
X	dimensionless coordinate
x, y, z	Cartesian coordinates [m]
Y	function defined in the text.

Greek symbols

α	modular angle [rad]
β	root of a cubic equation
Θ	angle defined in the text [rad]
λ	dimensional coefficient for the linear variation of internal heat generation [K^{-1}]
φ	amplitude [rad].

Subscripts

1, 2, 3	order of a root of a cubic equation
a	environment
av	average
b	on the y–z plane at $x = 0$
e	on the y–z plane at $x = U$
r	reduced value.

DIFFERENTIAL EQUATION OF TEMPERATURE DISTRIBUTION

The plate considered is shown in Fig. 1(a) together with the Cartesian coordinate system adopted. It is infinitely long in the y- and z-directions. Internal heat generation in it, Q, and the thermal conductivity of its material, K, depend on its temperature t linearly. The surface of the plate at $x = 0$ is thermally insulated and consequently the temperature t_b there is constant. The heat generated in the plate is dissipated to the surroundings at its remaining surface. The thickness of the plate, U, is constant. The temperature of the surroundings is equal to t_a. This temperature, t_b and U are known. For the assumptions made, the differential equation of the steady-state one-dimensional temperature distribution in the plate is given by

$$\frac{d}{dx}\left(K\frac{dt}{dx}\right) = -Q \tag{1}$$

where

$$K = K_a[1+n(t-t_a)] \tag{2}$$

$$Q = Q_a[1+\lambda(t-t_a)]. \tag{3}$$

The boundary conditions are expressed in

$$t = t_b \quad \text{for} \quad x = 0 \tag{4}$$

and

$$\frac{dt}{dx} = 0 \quad \text{for} \quad x = 0. \tag{5}$$

The values of K_a and Q_a in equations (2) and (3) are the thermal conductivity of the material of the plate and the internal heat generation in the plate at the temperature t_a. Both n and λ are the coefficients which characterize linear variation of K and Q, respectively. The values of K_a, Q_a, n and λ are known.

Equations (1)–(5) also apply to the plate shown in Fig. 1(b), provided that the temperature distribution in it is symmetrical with respect to its y–z plane at $x = 0$ and that the temperature on this plane is constant. It is then sufficient to determine the temperature distribution in one-half of the plate. To this end, the positive direction of the x-coordinate is considered.

Equation (1) will be transformed into a dimensionless form before solving it. For this purpose first equations (2) and (3) are reduced to

$$K = K_a(1+hT) \tag{6}$$

$$Q = Q_a(1+sT) \tag{7}$$

where T is the dimensionless temperature, and h and s the dimensionless coefficients which characterize the linear variation of the thermal conductivity of the material of the plate and internal heat generation in it, respectively. These are expressed as

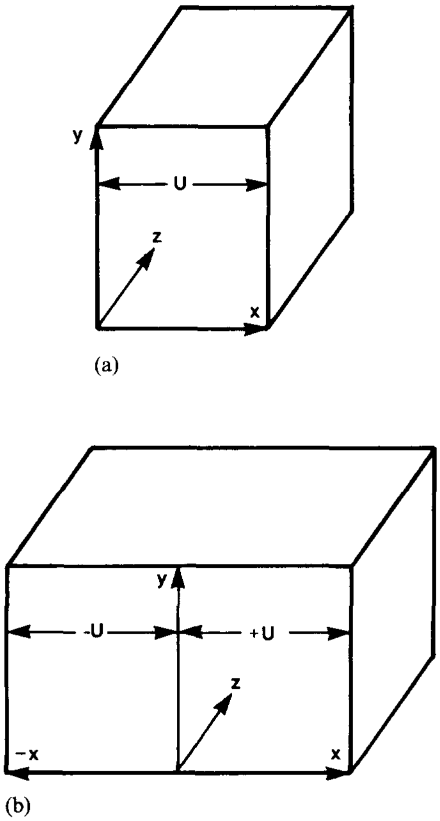

FIG. 1. Plate configurations considered: (a) plate of type a; (b) plate of type b.

$$T = \frac{t - t_a}{t_b - t_a} \quad (8)$$

$$h = \frac{K_b - K_a}{K_a} = n(t_b - t_a) \quad (9)$$

$$s = \frac{Q_b - Q_a}{Q_a} = \lambda(t_b - t_a). \quad (10)$$

K and Q given by equations (6) and (7) are inserted in equation (1) and the left-hand side of this equation is differentiated, and after rearrangement yields the differential equation of the temperature distribution in the plate

$$(1+hT)\frac{d^2T}{dX^2} + h\left(\frac{dT}{dX}\right)^2 = -\frac{Q_a U^2(1+sT)}{K_a(t_b - t_a)} \quad (11)$$

where X is the dimensionless coordinate. It is defined by

$$X = \frac{x}{U}. \quad (12)$$

In equations (11) and (12) U is the thickness of the plate shown in Fig. 1(a) and the half thickness of the plate is shown in Fig. 1(b). The boundary conditions in dimensionless form are given by

$$T = 1 \quad \text{for} \quad X = 0 \quad (13)$$

and

$$\frac{dT}{dX} = 0 \quad \text{for} \quad X = 0. \quad (14)$$

All quantities in equation (11) are known with the exception of T and its first and second derivatives.

SOLUTION OF DIFFERENTIAL EQUATION

Equation (11) is a second-order non-linear differential equation. In order to solve it, first it will be reduced to a first-order non-linear differential equation and thereafter the reduced equation will be integrated.

In order to reduce equation (11) to a first-order differential equation, a variable P is introduced such that

$$P = \frac{dT}{dX}. \quad (15)$$

Taking the derivatives of both sides of equation (15) with respect to X yields

$$P\frac{dP}{dT} = \frac{d^2T}{dX^2}. \quad (16)$$

The first and second derivatives of T in equation (11) are replaced by P and $P(dP/dT)$ (which are given by equations (15) and (16), respectively). After rearrangement, equation (11) becomes

$$\frac{dP}{dT} + \frac{hP}{1+hT} = \frac{W}{P}\frac{1+sT}{1+hT} \quad (17)$$

where

$$W = -\frac{Q_a U^2}{K_a(t_b - t_a)}. \quad (18)$$

Equation (17) is Bernoulli's equation. First it is reduced to a linear first-order differential equation and thereafter the latter is solved, as shown in a textbook on mathematics [3]. Omitting the details, the solution of equation (17) is presented below

$$\frac{dT}{dX} = -\left[\frac{2W}{(1+hT)^2}\left(\frac{sh}{3}T^3 + \frac{s+h}{2}T^2 + T\right) + \frac{C}{(1+hT)^2}\right]^{0.5}. \quad (19)$$

The negative sign in equation (19) is due to the fact that the temperature in the plate decreases along the x-coordinate. The integration constant C in this equation is determined using the boundary condition expressed in equation (14) and noting that $T = 1$ for $X = 0$. After the determination of this constant, the rearrangement of equation (19) yields

$$\frac{dT}{dX} = -\frac{1}{1+hT}(AY)^{0.5} \tag{20}$$

where

$$A = \frac{2Wsh}{3} = -\frac{2Q_a U^2 sh}{3K_a(t_b - t_a)} \tag{21}$$

$$Y = (T-1)\left[T^2 + \left(1 + \frac{3}{2}\frac{s+h}{sh}\right)T + 1 + \frac{3}{2}\frac{s+h}{sh} + \frac{3}{sh}\right]. \tag{22}$$

In order to integrate equation (20), it is reduced to

$$\frac{dT}{(AY)^{0.5}} + h\frac{T\,dT}{(AY)^{0.5}} = -dX. \tag{23}$$

Integration of the right-hand side of equation (23) is a straightforward matter. For the integration of the left-hand side of this equation, the roots of the cubic equation Y in it should be known. One of the roots of this cubic equation is equal to 1. The remaining two roots are found by solving the quadratic equation in equation (22) since the values of h and s are known. The algebraic sign of h and s is identical to that of n and λ, respectively, since $(t_b - t_a)$ is always positive (see equations (9) and (10)). Two conditions should be analysed, the condition that the cubic equation has three real roots and the condition that the cubic equation has one real and two complex roots [4].

The condition that the cubic equation has three real roots

Let the roots of equation (22) fulfil the requirement that $\beta_1 > \beta_2 > \beta_3$. For the dimensionless temperature in the plate, the following inequality holds: $T \leqslant 1$.

The integration of the left-hand side of equation (23) is defined in the intervals $T \geqslant \beta_1$ and $\beta_2 \geqslant T \geqslant \beta_3$ if $A > 0$ and in the intervals $\beta_1 \geqslant T \geqslant \beta_2$ and $\beta_3 \geqslant T$ if $A < 0$ [4]. For $h > 0$ and $s > 0$, and for $h < 0$ and $s < 0$, A in equation (23) is always negative (see equation (21)). For $h > 0$ and $s < 0$, and for $h < 0$ and $s > 0$, A is always positive. The foregoing implies three cases.

Case (a). If $\beta_1 > \beta_2 > \beta_3 = 1$, the integration of the left-hand side of equation (23) is defined for $A < 0$ (i.e. for $h > 0$ and $s > 0$, and for $h < 0$ and $s < 0$) since $T \leqslant 1$; thus this integration can be carried out in order to determine the dimensionless temperature distribution in the plate for $A < 0$. If $A > 0$ (i.e. for $h > 0$ and $s < 0$, and for $h < 0$ and $s > 0$), the foregoing integration is not defined. This means that the plate fails to operate for given values of h and s (or for those of t_b, t_a, n and λ); in other words, the temperature distribution in the plate is incalculable for these values of h and s.

Case (b). If $\beta_1 > \beta_2 = 1 > \beta_3$, the integration of the left-hand side of equation (23) can be only carried out for $A > 0$. If $A < 0$, this integration is not defined.

Case (c). If $\beta_1 = 1 > \beta_2 > \beta_3$, the integration of the left-hand side of equation (23) can be carried out only for $A < 0$. If $A > 0$, this integration is not defined.

The cases dealt with above are summarized in Table 1. An equation number given in this table indicates the existing solution.

The integration of equation (23) is given with equations (24a)–(24c), respectively [4]. Further on a-versions of the equations will be valid for Case (a); b-versions for Case (b) and c-versions for Case (c). An equation numbered without a letter applies to all three cases

$$W_3F(\varphi/\alpha) + W_3h[\beta_3F(\varphi/\alpha) + W_4E(\varphi/\alpha) + W_4W_1\cot\varphi] = -X + C \tag{24a}$$

$$W_3F(\varphi/\alpha) + W_3h[\beta_1F(\varphi/\alpha) - W_4E(\varphi/\alpha)] = -X + C \tag{24b}$$

$$W_3F(\varphi/\alpha) + W_3h[\beta_3F(\varphi/\alpha) + W_4E(\varphi/\alpha) - W_5W_2] = -X + C \tag{24c}$$

where

$$W_1 = (1 - k^2\sin^2\varphi)^{0.5} \tag{25a}$$

$$W_2 = \frac{\sin\varphi\cos\varphi}{(1 - k^2\sin^2\varphi)^{0.5}} \tag{26c}$$

$$\varphi = \arcsin\left[\left(\frac{b - Td}{Tc - a}\right)^{0.5}\right] \quad \text{for} \quad 0 \leqslant \varphi \leqslant \pi/2 \tag{27}$$

$$\alpha = \arcsin|k| \tag{28}$$

$$a = \beta_1 \tag{29a}$$

$$a = \beta_2 - \beta_3 \tag{29b}$$

$$a = -\beta_3 k^2 = -\beta_3\frac{\beta_1 - \beta_2}{\beta_1 - \beta_3} \tag{29c}$$

$$b = \beta_3 - \beta_1 \tag{30a}$$

$$b = \beta_3 \tag{30b}$$

$$b = \beta_2 \tag{30c}$$

$$c = 1 \tag{31a}$$

$$c = 0 \tag{31b}$$

$$c = -k^2 = -\frac{\beta_1 - \beta_2}{\beta_1 - \beta_3} \tag{31c}$$

$$d = 0 \tag{32a}$$

$$d = 1 \tag{32b, c}$$

$$k^2 = \frac{\beta_1 - \beta_2}{\beta_1 - \beta_3} \tag{33a, c}$$

$$k^2 = \frac{\beta_2 - \beta_3}{\beta_1 - \beta_3} \tag{33b}$$

$$W_3 = 2/[|A|(\beta_1 - \beta_3)]^{0.5} \tag{34}$$

$$W_4 = \beta_1 - \beta_3 \tag{35}$$

Table 1. Conditions for the existence of a solution for equation (23)

The signs of h and s	The sign of A	Case (a) $\beta_1 > \beta_2 > \beta_3 = 1$	Three roots are real Case (b) $\beta_1 > \beta_2 = 1 > \beta_3$	Case (c) $\beta_1 = 1 > \beta_2 > \beta_3$	Two roots are complex Case (d) $\beta_1 = 1$
$h > 0$ and $s > 0$	$-$	equation (24a)	no solution	equation (24c)	no solution
$h > 0$ and $s < 0$	$+$	no solution	equation (24b)	no solution	no solution
$h < 0$ and $s > 0$	$+$	no solution	equation (24b)	no solution	no solution
$h < 0$ and $s < 0$	$-$	equation (24a)	no solution	equation (24c)	equation (24d)

$$W_5 = \beta_1 - \beta_2. \tag{36}$$

In equation (24) $F(\varphi/\alpha)$ and $E(\varphi/\alpha)$ are Legendre's normal elliptic integrals of the first and second kinds. Their values are determinable if the amplitude φ and modular angle α in them are known. α is a function of the roots of the foregoing cubic equation and thus it is known. φ is a function of these roots and T. For a given value of T, φ is known.

In order to determine the integration constant C in equation (24), the boundary condition expressed in equation (13) is used. Then this constant is given by

$$W_3F(\varphi_b/\alpha) + W_3h[\beta_3F(\varphi_b/\alpha) + W_4E(\varphi_b/\alpha) + W_4(W_1)_b \cot\varphi_b] = C \tag{37a}$$

$$W_3F(\varphi_b/\alpha) + W_3h[\beta_1F(\varphi_b/\alpha) - W_4E(\varphi_b/\alpha)] = C \tag{37b}$$

$$W_3F(\varphi_b/\alpha) + W_3h[\beta_3F(\varphi_b/\alpha) + W_4E(\varphi_b/\alpha) - W_5(W_2)_b] = C \tag{37c}$$

where φ_b, the value of φ for $X = 0$, is determined from equation (27). For this purpose the values of T, a, b, c and d given by equations (13) and (29)–(32) are inserted in equation (27). After rearrangement and replacing 1 in the rearranged equation with the proper root (i.e. $1 = \beta_3$; $1 = \beta_2$ and $1 = \beta_1$ for Cases (a), (b) and (c), respectively) the equation yields

$$\varphi_b = \arcsin 1 = \pi/2. \tag{38}$$

Both W_1 in equation (37a) and W_2 in equation (37c) should be evaluated using the value of φ_b given by equation (38), as already indicated in these equations.

For given values of t_b, t_a, n, λ, U, K_a and Q_a, all the quantities in equation (24) are known with the exception of the dimensionless temperature T and dimensionless coordinate X. This equation is an implicit function of T. How the value of T can be determined for a given value of X will be explained after the next section.

The condition that the cubic equation has one real and two complex roots

For this condition the integration of equation (23) is given with equation (24d) [4]. Further on the equations analogous to those given in the condition studied previously are considered d-versions of the latter

$$W_6F(\varphi/\alpha) + h[W_7 - W_8F(\varphi/\alpha) + W_9E(\varphi/\alpha)] = -X + C \tag{24d}$$

where

$$W_6 = -\frac{a+b}{W_{10}} \tag{39}$$

$$W_7 = \frac{(a+b)^2}{W_{10}}\frac{1+\cos\varphi}{\sin\varphi}(1-k^2\sin^2\varphi)^{0.5} \tag{40}$$

$$W_8 = \frac{(a+b)b}{W_{10}} \tag{41}$$

$$W_9 = \frac{(a+b)^2}{W_{10}} \tag{42}$$

$$\varphi = \arccos\left(\frac{b-Td}{Tc-a}\right) \quad \text{for} \quad 0 < \varphi \leqslant \pi \tag{27d}$$

$$\alpha = \arcsin k \tag{28d}$$

$$W_{10} = \left(\frac{4AS^3}{\sin^3 2\Theta}\right)^{0.5} \tag{43}$$

$$a = -r + S\tan\Theta \tag{29d}$$

$$b = r + S\cot\Theta \tag{30d}$$

$$c = -1 \tag{31d}$$

$$d = 1 \tag{32d}$$

$$k = |\sin\Theta| \tag{33d}$$

$$\tan 2\Theta = \frac{S}{1-r}\begin{cases} 0 < 2\Theta < \pi & \text{for} \quad T \geqslant 1 (A > 0) \\ -\pi < 2\Theta < 0 & \text{for} \quad T \leqslant 1 (A < 0) \end{cases} \tag{44}$$

$$S = \left(1 + \frac{3}{2}\frac{s+h}{sh} + \frac{3}{sh} - r^2\right)^{0.5} \quad (S > 0) \tag{45}$$

$$r = -\frac{1}{2}\left(1 + \frac{3}{2}\frac{s+h}{sh}\right). \tag{46}$$

In order to determine the integration constant C in equation (24d), the boundary condition expressed in equation (13) is used. This constant is given by

$$W_6F(\varphi_b/\alpha)+h[(W_7)_b-W_8F(\varphi_b/\alpha)+W_9E(\varphi_b/\alpha)] = C. \quad (37d)$$

In equation (37d), φ_b, the value of φ at $X = 0$, is determined with equation (27d). To this end, the values of T, a, b, c and d given by equations (13) and (29d)–(32d) are inserted in equation (27d). After rearrangement, the resulting expression $(1-r)$ both in the numerator and denominator of this equation is replaced by $S/(\tan 2\Theta)$, as calculated with equation (44). If $\tan 2\Theta$ and $\cot \Theta$ are expressed as functions of $\tan \Theta$ in the rearranged equation, this equation produces the value of φ_b after rearrangement as

$$\varphi_b = \arccos -1 = \pi. \quad (38d)$$

Equation (24d) is valid only for $h < 0$ and $s < 0$. For the remaining combinations of h and s, the integration of equation (23) is not defined, as indicated in Table 1, and equation (24d) is invalid. This is explained below.

If $h > 0$ and $s > 0$, the value of A is negative (see equation (21)). For $A < 0$, the value of Θ defined by equation (44) is also negative. Since the value of r given by equation (46) is negative for $h > 0$ and $s > 0$ and the value of S expressed in equation (45) is per definition positive, the value of $S/(1-r)$ in equation (44) is positive. A positive value of $S/(1-r)$ yields a positive value for Θ, which is in contradiction to the definition of Θ. Thus for $h > 0$ and $s > 0$, the integration of equation (23) is not defined.

If $h > 0$ and $s < 0$ and if $h < 0$ and $s > 0$, the value of A is positive. It follows from equation (44) that T should be equal to 1 or greater than 1 if $A > 0$, which is in contradiction to the temperature distribution in the plate analysed (i.e. $T \leqslant 1$). Thus the integration of equation (23) is not defined for the aforesaid values of h and s.

For given values of t_b, t_a, n, λ, U, K_a and Q_a, all the quantities in equation (24d) are known with the exception of the dimensionless temperature T and dimensionless coordinate X; thus this equation can be used to determine the value of T for a given value of X for $h < 0$ and $s < 0$, as explained below.

Calculation of dimensionless temperature

As stated earlier, the following are assumed to be known for the plate dealt with: t_b, the temperature of the plane of the plate at $x = 0$; t_a, the temperature of the surroundings; U, the thickness or half of the thickness of the plate; n and λ, the dimensional coefficient for the linear variation of the thermal conductivity of the material of the plate and internal heat generation in the plate, respectively; and K_a and Q_a, the thermal conductivity of the material of the plate and internal heat generation in the plate at the temperature t_a, respectively.

In order to determine the value of T for a given value of X, equation (24) should be used and this equation is the solution of equation (23). However, this solution does not exist for some combinations of the values of t_b, t_a, n and λ as Table 1 implies (i.e. $h = n(t_b-t_a)$ and $s = \lambda(t_b-t_a)$). Therefore, before using equation (24), this equation should be checked whether it is a solution of equation (23). To this end first h and s given by equations (9) and (10) are determined, and thereafter the roots of the quadratic equation in equation (22). These roots depend on the values of h and s or those of t_b, t_a, n and λ but not those of U, Q_a and K_a. Table 1 is consulted whether there is a solution for equation (23) for these known values of h and s. If there is no solution for equation (23), the plate dealt with fails to operate for given values of t_b, t_a, n and λ. If it is desired to make the plate operational, there seems to be no other way than to vary the value of t_b (i.e. a boundary value) since the values of λ and n are fixed for a given plate and t_a, and the existence of a solution for equation (23) is independent of the value of t_a, as will be explained in the next section. Consequently the value of t_b is varied till the roots of the foregoing quadratic equation fulfil one of the conditions given in Table 1 for the existence of a solution for equation (23). If this is not possible for all possible values of t_b, the operation of the plate is invalid.

If there is a solution for equation (23) for given values of t_b, n and λ, then this solution is given by equation (24), and h, s and the roots of the quadratic equation in equation (22) are already known. Using the values of these roots, h, s, t_a, U, K_a, Q_a and X, all the quantities in equation (24) can be calculated with the exception of T (or φ), as equation (21) and equations (38)–(25) or equations (38d)–(39) imply. The foregoing quantities should be successively evaluated starting with equation (38) or equation (38d). In order to solve T from equation (24) for a given X, a value is assumed for T (i.e. $T < 1$), and φ is determined with equation (27). The value of α is already known. With known values of φ and α, $F(\varphi/\alpha)$ and $E(\varphi/\alpha)$ are determined either from the tables given for $F(\varphi/\alpha)$ and $E(\varphi/\alpha)$ [5] or with the analytic expressions of $F(\varphi/\alpha)$ and $E(\varphi/\alpha)$ [4]. The known values of $F(\varphi/\alpha)$, $E(\varphi/\alpha)$, X and the remaining relevant quantities are inserted in equation (24). The value of T is iterated till equation (24) is satisfied to within the desired accuracy.

A computer program seems to make it convenient to calculate the temperature distribution in a given plate. The relations required for such a program and simple formulae for the prediction of $E(\varphi/\alpha)$ are given in the Appendix for the convenience of the practising engineer. Simple formulae for the determination of $F(\varphi/\alpha)$ are given in ref. [6].

DISCUSSION OF RESULTS

It follows from the previous section that the temperature distribution in a plate with linearly temperature-dependent thermal conductivity and internal

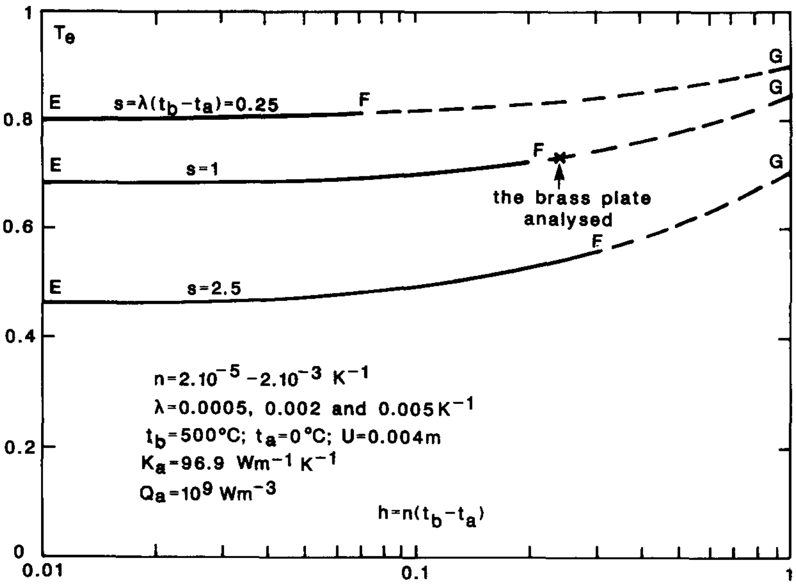

FIG. 2. Dimensionless temperature in a plate vs the dimensionless coefficient for the linear variation of the thermal conductivity of the material of the plate.

heat generation cannot be determined with some combinations of the values of t_b, n and λ. Now a few examples for these combinations will be illustrated.

Consider the plate shown in Fig. 1(a). Let $U = 0.004$ m; $t_b = 500°C$; $t_a = 0°C$; $K_a = 96.9$ W m^{-1} K^{-1} and $Q_a = 10^9$ W m^{-3}. Three values for λ were taken into account: 0.0005, 0.002 and 0.005 K^{-1}. For these values of λ, the values of s are 0.25, 1 and 2.5 (i.e. $s = \lambda t_b$). Keeping the values of t_b, t_a, U, K_a, Q_a and s constant, T_e, the dimensionless temperature on the y–z plane of the plate at $X = 1$ (i.e. $x = U$) was determined with equation (24) for a given value of h (i.e. $h = nt_b$). n was varied between 0.00002 and 0.002 K^{-1} and consequently h between 0.01 and 1. In Fig. 2 the dimensionless temperatures calculated are shown as a function of h, s being a parameter. The full lines between E and F on the curves in this figure give the dimensionless temperatures obtained with equation (24).

Consider the curve for $s = 2.5$ in Fig. 2. If h varies between 0.01 and 0.3124, equation (24) yields a value for T_e (i.e. $\beta_1 = 1 > \beta_2 > \beta_3$). If h varies between 0.3125 and 1, the quadratic equation in equation (22) has one real and two complex roots and the temperature distribution in the plate is not defined (i.e. indeterminable) as can be deduced from Table 1 (i.e. $h > 0$ and $s > 0$). If h is increased with sufficiently small increments, its initial value being 0.3124, it gradually approaches to 0.3125, φ and α in equation (24) to $\pi/2$ and $F(\varphi/\alpha)$ to infinity. Thus there exists a value of h between 0.3124 and 0.3125 beyond which the temperature in the plate is incalculable. The limit value of T_e corresponds to the foregoing value of h. The calculations have been carried out on a personal computer with double precision.

Equation (24) yields a value for T_e for $0.01 \leqslant h \leqslant 0.2000$ and $0.01 \leqslant h \leqslant 0.07142$ if $s = 1$ and 0.25, respectively, as shown in Fig. 2. The temperature distribution in the plate is not defined (i.e. indeterminable) for $0.2001 \leqslant h \leqslant 1$ if $s = 1$ and for $0.07143 \leqslant h \leqslant 1$ if $s = 0.25$.

The existence of a solution for equation (23) does not depend on the values of U, K_a and Q_a since the roots of the quadratic equation in equation (22) are only functions of the values of t_b, t_a, n and λ. This existence is also independent of the value of t_a, which can be tentatively shown by evaluating the temperature distribution in the plate with different values of t_a for given values of t_b, U, K_a, Q_a, n and λ. For instance, for $U = 0.004$ m, $t_b = 500°C$, $t_a = 0°C$, $K_a = 96.9$ W m^{-1} K^{-1}, $Q_a = 10^9$ W m^{-3}, $n = 0.0003$ K^{-1} and $\lambda = 0.002$ K^{-1}, t_e, the temperature of the plane of the plate at $X = 1$ (i.e. $x = U$) is equal to 357.1926°C. The foregoing values of K_a, Q_a, n and λ are valid for $t_a = 0°C$. If t_a is different from 0°C, the values of K_a, Q_a, n and λ are also different from those for $t_a = 0°C$, as equations (2) and (3) imply. Taking this into consideration and keeping the values of t_b and U constant, t_e was determined with different values of t_a. All the integer values of t_a were taken into account between 357 and −273°C. For all the values of t_a considered, t_e was equal to that calculated for $t_a = 0$ (i.e. accurate to 12 decimal places).

The quadratic equation in equation (22) dictates whether there is a solution for the temperature distribution in the plate analysed. In this quadratic equation, the values of h and s are interchangeable, i.e. the value of h can be replaced by that of s and vice versa. In such a case, the quadratic equation yields the same roots. Thus taking $h = 0.25$, 1 and 2.5 and keeping the values of t_b, t_a, U, K_a, Q_a and h constant, s was varied between 0.01 and 1, and the values of T_e were

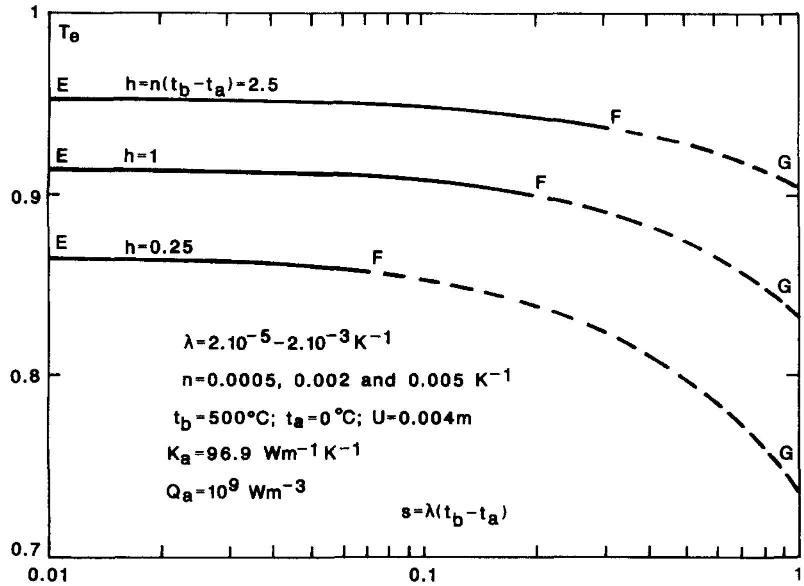

FIG. 3. Dimensionless temperature in a plate vs the dimensionless coefficient for the linear variation of internal heat generation in the plate.

determined as a function of s, h being a parameter. The results obtained are shown in Fig. 3, which are in fact analogous to those shown in Fig. 2. The EF parts of the curves drawn with full lines in Fig. 3 give the values of T_e obtained with equation (24). This equation yields a value for T_e for $0.01 \leqslant s \leqslant 0.3124$, $0.01 \leqslant s \leqslant 0.2000$ and $0.01 \leqslant s \leqslant 0.07142$ for $h = 2.5$, 1 and 0.25, respectively. The temperature distribution in the plate is not defined for $0.3125 \leqslant s \leqslant 1$, $0.2001 \leqslant s \leqslant 1$ and $0.07143 \leqslant s \leqslant 1$ for $h = 2.5$, 1 and 0.25, respectively.

The assumptions that the thermal conductivity of the material of the plate and internal heat generation in the plate vary linearly for a quite large temperature range seem probably to be an exaggeration. In so far as the possibility of the solution of equation (23) is concerned, this exaggeration is not of much practical significance as can be deduced from Fig. 3 (i.e. $428.32 \leqslant t_e(°C) \leqslant 475.76$) and from the fact that the existence of the solution of equation (23) is not dependent on the value of t_a. This value can be always assumed to be in the vicinity of the value of t_e.

For the conditions considered in Figs. 2 and 3, the values of T_e were also determined with a numerical (i.e. Runge–Kutta) method using equation (11). The results obtained are shown in Figs. 2 and 3. For the EF parts of the curves given in these figures, the values of T_e calculated with the numerical method practically coincide with those calculated with equation (24). The FG parts of the curves shown with dotted lines in these figures give also the values of T_e obtained with the numerical method, which are in fact trivial, as demonstrated earlier. Then it seems justifiable to conclude that the numerical method used fails to predict the temperature distribution in a plate for some combinations of the values of t_b, n and λ if the thermal conductivity of the material of the plate and internal heat generation in the plate depend on the temperature of the plate linearly.

The foregoing appears to be a foregone conclusion since it is dealt with by a physical phenomenon. If a numerical method is used to solve equation (11), the solution obtained is an approximation and consequently the thermal conductivity of the material of a plate and internal heat generation in the plate need not vary linearly with temperature. If an analytic method is used for the solution of equation (11), as carried out herein, the solution obtained is exact and the thermal conductivity of the material of the plate and internal generation in it do strictly depend on the temperature of the plate linearly. This is further illustrated below.

Taking the thermal conductivity of the material of a plate constant while internal heat generation in it varies linearly with its temperature, equation (1) has been solved yielding

$$T = \left(1 + \frac{1}{s}\right)\cos(W_{11}^{0.5}X) - s^{-1} \quad \text{for} \quad s > 0 \quad (47)$$

$$T = \left(1 + \frac{1}{s}\right)\cosh[(-W_{11})^{0.5}X] - s^{-1} \quad \text{for} \quad s < 0 \quad (48)$$

where

$$W_{11} = \frac{Q_a U^2 \lambda}{K_{av}}. \quad (49)$$

In equation (49) K_{av} is the average value of the thermal conductivity of the material of the plate and it is evaluated with equation (2) using the value of the arithmetic mean temperature of the plate (i.e. $t_{av} = (t_b + t_e)/2$).

Considering the conditions given in Figs. 2 and 3, the values of T_e were determined with equation (47). The values obtained for T_e are practically identical to those obtained with the numerical method (i.e. within 1.4% accuracy). If the thermal conductivity of the material of the plate and internal heat generation in the plate vary linearly with the temperature of the plate, equation (47) yields trivial solutions for $0.3125 \leqslant h \leqslant 1$, $0.2001 \leqslant h \leqslant 1$, and $0.07143 \leqslant h \leqslant 1$ if $s = 2.5$, 1 and 0.25, respectively, and for $0.3125 \leqslant s \leqslant 1$, $0.2001 \leqslant s \leqslant 1$ and $0.07143 \leqslant s \leqslant 1$ if $h = 2.5$, 1 and 0.25, respectively, similar to the solutions obtained with the numerical method.

The assumption that the thermal conductivity of the material of the plate is constant seems to be acceptable for the solution of equation (1) if the solution obtained with equation (47) is not trivial and if the relevant conditions given in Figs. 2 and 3 are taken into account. For the conditions considered in Figs. 2 and 3 for which a solution of equation (23) exists, the error in the prediction of T_e with equation (47) is within 0.25%.

A brass plate will now be analysed. For this plate $t_b = 500°C$, $t_a = 0°C$, $U = 0.004$ m and $Q_a = 10^9$ W m^{-3}. For brass and $t_a = 0$, the values of n, λ and K_a can be approximately taken equal to 0.00049 K^{-1}, 0.002 K^{-1} and 96.9 W m^{-1} K^{-1}, respectively. For the values of t_b, t_a, n and λ considered, the values of h and s become 0.245 and 1, respectively. It follows from Fig. 2 that the temperature distribution in the brass plate analysed is incalculable. This temperature distribution is also not defined for $500 > t_b(°C) \geqslant 270.41$. For $270.40 \geqslant t_b(°C) \geqslant 93.29$, this temperature distribution is determinable and $T_e = 0.5846$ for $t_b = 270.40°C$ and to 0.000348 for $t_b = 93.29°C$. The numerical method mentioned earlier and equation (47) yield values for T_e for the temperature range of $500 \geqslant t_b(°C) \geqslant 270.41$. For this temperature range, the temperature distribution in the brass plate is, in fact, not defined (i.e. incalculable) if the thermal conductivity of the material of the plate and internal heat generation in the plate vary linearly with temperature.

SUMMARY/CONCLUSIONS

The analytic solution of the differential equation for the steady-state one-dimensional temperature distribution in a plate with linearly temperature-dependent thermal conductivity and internal heat generation is presented. This equation is a non-linear second-order differential equation and its solution is not available in the literature. The boundary conditions assumed are that the temperature at the base of the plate is constant and that the plate is thermally insulated there. This solution is an implicit function of the temperature and it includes, among other things, Legendre's normal elliptic integrals of the first and second kinds. The determination of these elliptic integrals are explained for the convenience of the practising engineer.

The solution of the foregoing differential equation is only defined for some combinations of the values of the base temperature of the plate and the coefficients which characterize linear variations of the thermal conductivity of the material of the plate and internal heat generation in the plate. It is shown that the solution of this differential equation with a numerical method and its analytic solution with the assumption that the thermal conductivity of the material of the plate is constant while this thermal conductivity depends on the temperature of the plate linearly are trivial for some combinations of the values of the base temperature of the plate and the coefficients which characterize linear variations of the thermal conductivity of the material of the plate and internal heat generation in the plate. Simple criteria are presented to avoid these trivial solutions.

The differential equation dealt with is also of practical significance in chemical and nuclear engineering.

Acknowledgement—This is a government sponsored work carried out by TNO.

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APPENDIX: SIMPLE FORMULAE FOR THE DETERMINATION OF $E(\varphi/\alpha)$

The values of $F(\varphi/\alpha)$ and $E(\varphi/\alpha)$ are tabulated as functions of φ and α for $0 < \varphi \leqslant \pi/2$ and $0 < \alpha \leqslant \pi/2$ [5]. The value of φ can vary between 0 and π and that of α between 0 and $\pi/2$. If the tabulated values are used to determine $F(\varphi/\alpha)$ and $E(\varphi/\alpha)$ for $\varphi > \pi/2$, first $F[(\pi-\varphi)/\alpha]$, $F[(\pi/2)/\alpha]$, $E[(\pi-\varphi)/\alpha]$ and $E[(\pi/2)/\alpha]$ should be determined from the tabulated values and thereafter $F(\varphi/\alpha)$ and $E(\varphi/\alpha)$ with equations (A1) and (A2), respectively

$$F(\varphi/\alpha) = 2F[(\pi/2)/\alpha] - F[(\pi-\varphi)/\alpha] \quad \text{(A1)}$$

$$E(\varphi/\alpha) = 2E[(\pi/2)/\alpha] - E[(\pi-\varphi)/\alpha]. \quad \text{(A2)}$$

Simple formulae for the determination of $F(\varphi/\alpha)$ are established in ref. [6]. However, one of the formulae given in ref. [6] is incorrect. J_1 defined by equation (A2) in ref. [6] should be equal to sin α but not to arcsin α.

Herein simple formulae for the calculation of $E(\varphi/\alpha)$ are presented. Legendre's normal elliptic integral of the second kind, $E(\varphi/\alpha)$, is an infinite series, which does not rapidly converge for large values of α. For the adequate prediction of $E(\varphi/\alpha)$ (i.e. to obtain the tabulated values of $E(\varphi/\alpha)$) it is sufficient to consider the first 4, 9 and 46 terms in this infinite series for $\alpha = \pi/12$, $3\pi/12$ and $5\pi/12$, respectively.

For $0 \leqslant \varphi \leqslant \pi$ and $0 \leqslant \alpha \leqslant \pi/12$, $E(\varphi/\alpha)$ is given by

$$E(\varphi/\alpha) = \varphi - J_1^2(0.5J_2 + 0.125J_1^2J_3 + 0.0625J_1^4J_4) \quad \text{(A3)}$$

where

$$J_1 = \arcsin k = \sin\alpha \quad \text{(A4)}$$

$$J_2 = 0.5(\varphi - \sin\varphi\cos\varphi) \quad \text{(A5)}$$

$$J_3 = 0.75(J_2 - \tfrac{1}{3}\sin^3\varphi\cos\varphi) \quad \text{(A6)}$$

$$J_4 = \tfrac{5}{6}(J_3 - 0.2\sin^5\varphi\cos\varphi). \quad \text{(A7)}$$

Equation (A3) yields the tabulated values of $E(\varphi/\alpha)$ for the foregoing ranges of φ and α.

If $0 < \varphi \leqslant \pi/2$ and $\pi/12 < \alpha \leqslant \pi/2$, the values of α and φ should be reduced till $\alpha \leqslant \pi/12$. In this case, equation (A3) can be used to determine $E(\varphi/\alpha)$. For this reduction the Gauss transformation is used [4]. Let

$$k = \sin\alpha \quad \text{(A8)}$$

$$KU = (1-k^2)^{0.5} \quad \text{(A9)}$$

$$K_1 = \frac{1-KU}{1+KU}. \quad \text{(A10)}$$

The reduced values of the amplitude and the modular angle are given by

$$\varphi_r = \arcsin\left(\frac{1-(1-k^2\sin^2\varphi)^{0.5}}{(1-KU)\sin\varphi}\right) \quad \text{(A11)}$$

$$\alpha_r = \arcsin K_1 \quad \text{(A12)}$$

and the following relation holds between $E(\varphi/\alpha)$ and $E(\varphi_r/\alpha_r)$

$$E(\varphi/\alpha) = \frac{2K_1}{1+K_1}\,\frac{\sin\varphi_r\cos\varphi_r(1-K_1^2\sin^2\varphi_r)^{0.5}}{1+K_1\sin^2\varphi_r} - (1-K_1)F(\varphi_r/\alpha_r) + \frac{2}{1+K_1}E(\varphi_r/\alpha_r). \quad \text{(A13)}$$

If the reduced modular angle is still higher than $\pi/12$, then the values of α_r and φ_r are successively reduced using equations (A8)–(A13) until the last reduced modular angle is equal to $\pi/12$ or smaller than $\pi/12$, for which equation (A3) is valid.

If $\pi/2 < \varphi \leqslant \pi$ and $\pi/12 < \alpha \leqslant \pi/2$, $E(\varphi/\alpha)$ is divided into two parts using equation (A2). Since $(\pi - \varphi)$ is smaller than $\pi/2$, the previously described method is used to determine $E[(\pi-\varphi)/\alpha]$ and $E[(\pi/2)/\alpha]$, and $E(\varphi/\alpha)$ is predicted with equation (A2).

For the calculations carried out in the present study for the condition that the cubic equation Y in equation (23) has three real roots, three successive reductions of the modular angle and amplitude were sufficient. The maximum value of the unreduced modular angle obtained for these calculations was about 89.84 $\pi/180$ rad. Three successive reductions of this maximum modular angle yielded a reduced modular angle less than $\pi/12$ rad.

For the condition that the aforesaid cubic equation has one real and two complex roots, the maximum value of the modular angle appears to be $\pi/12$ rad. Thus $E(\varphi/\alpha)$ can be predicted with equation (A3) and $F(\varphi/\alpha)$ with equation (A1) in ref. [6] without reducing the values of the modular angle and amplitude. The foregoing can be tentatively shown by calculating the modular angle for $-1 < h \leqslant -0.001$ and $-1 < s \leqslant -0.001$ and considering each value of h or s to three decimal places. The minimum value of h or s is greater than -1, as equation (6) or (7) implies. (Note that $h < 0$ and $s < 0$.)

DISTRIBUTION DE TEMPERATURE DANS UNE PLAQUE AVEC CONDUCTIVITE THERMIQUE DEPENDANT DE LA TEMPERATURE ET GENERATION INTERNE DE CHALEUR

Résumé—On présente la solution analytique de l'équation différentielle, d'état permanent, de la distribution monodimensionnelle de température dans une plaque avec conductivité thermique variable en fonction de la température et avec génération interne de chaleur. On montre que la résolution de cette équation par une méthode numérique et la résolution analytique avec l'hypothèse d'une conductivité du matériau est constante tandis qu'elle dépend linéairement de la température, sont triviales pour quelques conditions qui peuvent être importantes dans des applications pratiques. Des critères simples sont donnés pour éviter ces solutions triviales.

TEMPERATURVERTEILUNG IN EINER PLATTE MIT TEMPERATURABHÄNGIGER WÄRMELEITFÄHIGKEIT UND INNEREN WÄRMEQUELLEN

Zusammenfassung—Die Differentialgleichung für die stationäre eindimensionale Temperaturverteilung in einer Platte wird gelöst. Die lineare Temperaturabhängigkeit der Wärmeleitfähigkeit sowie das Auftreten von inneren Wärmequellen wird dabei berücksichtigt. Unter bestimmten Bedingungen ergeben sich triviale Lösungen der Differentialgleichung sowohl bei der numerischen als auch der analytischen Berechnung. Bei der analytischen Berechnung wird die Wärmeleitfähigkeit als konstant angenommen, während beim numerischen Verfahren die Temperaturabhängigkeit der Wärmeleitfähigkeit berücksichtigt wird. Anhand einfacher Kriterien läßt sich die Berechnung der trivialen Lösungen vermeiden, dies ist wichtig für praktische Anwendungen.

РАСПРЕДЕЛЕНИЕ ТЕМПЕРАТУР В ПЛАСТИНЕ С ЗАВИСЯЩИМИ ОТ ТЕМПЕРАТУРЫ ТЕПЛОПРОВОДНОСТЬЮ И ВНУТРЕННИМ ТЕПЛОВЫДЕЛЕНИЕМ

Аннотация—Представлено аналитическое решение дифференциального уравнения стационарного одномерного распределения температур в пластине с линейно зависящими от температуры теплопроводностью и внутренним тепловыделением. Показано, что численное решение данного дифференциального уравнения, а также его аналитическое решение для случаев постоянного коэффициента теплопроводности материала пластины и его линейной зависимости от температуры в некоторых условиях являются тривиальными, что может иметь существенное значение для практических целей. Предложены простые критерии, позволяющие исключить такие тривиальные решения.